

PRIME GEODESIC THEOREM OF GALLAGHER TYPE

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ABSTRACT. We reduce the exponent in the error term of the prime geodesic theorem for compact Riemann surfaces from $\frac{3}{4}$ to $\frac{7}{10}$ outside a set of finite logarithmic measure.

1. INTRODUCTION

Let $\Gamma \subset PSL(2, \mathbb{R})$ be a strictly hyperbolic Fuchsian group and $\mathcal{F} = \Gamma \backslash \mathcal{H}$ be the corresponding compact Riemann surface of a genus $g \geq 2$, where $\mathcal{H} = \{z = x + iy : y > 0\}$ denotes the upper half-plane equipped with the hyperbolic metric $ds^2 = \frac{dx^2 + dy^2}{y^2}$. The Selberg zeta function on \mathcal{F} is defined by the Euler product

$$Z(s) = Z_\Gamma(s) = \prod_{\{P_0\}} \prod_{k=0}^{\infty} (1 - N(P_0)^{-s-k}), \quad \operatorname{Re}(s) > 1,$$

where the product is taken over all primitive conjugacy classes $\{P_0\}$ and $N(P_0)$ is the norm of a conjugacy class P_0 . (See any of the standard references [9], [13] for a necessary background.)

The Selberg zeta function can be continued to the whole complex plane as a meromorphic function of a finite order. Its zeros $\rho = \frac{1}{2} + i\gamma$ are denumerable and closely related to the eigenvalues λ of the Laplace-Beltrami operator on \mathcal{F} . This operator being essentially self-adjoint, its eigenvalues are non-negative and tend to infinity. Therefore, there are finitely many of them that are less than $\frac{1}{4}$. We have $\gamma = \pm\sqrt{\lambda - \frac{1}{4}}$ for $\lambda \geq \frac{1}{4}$ and $\gamma = \mp i\sqrt{\frac{1}{4} - \lambda}$ for $\lambda < \frac{1}{4}$. So, the zeros of Z are split into two parts: those lying on the critical line $\operatorname{Re} s = \frac{1}{2}$ and the real zeros in the interval $[0, 1]$.

Compared to the Riemann zeta case, there are "too many zeros" of Z in some sense [9, (6.14) on p. 113]. Let $N(t)$ denote the number of zeros $\rho = \frac{1}{2} + i\gamma$ such that $0 < \gamma \leq t$. The function $R(t)$, given by $N(t) = \frac{|\mathcal{F}|}{4\pi} t^2 + R(t)$, grows as $O(t(\log t)^{-1})$, where $|\mathcal{F}|$ is the volume of \mathcal{F} .

The norm $N(P_0)$ is determined by the length of the geodesic joining two fixed points, necessarily the same ones for all representatives of P_0 . The statement about the number $\pi_0(x)$ of classes $\{P_0\}$ such that $N(P_0) \leq x$, for $x > 0$, is known as the prime geodesic theorem. It has been proved by Selberg [21] and Huber [10, 11] and subsequently generalized to various settings. The references [16, 5, 8, 18, 6, 19] form an interesting range of samples.

2010 *Mathematics Subject Classification.* 11M36, 11F72, 58J50.

Key words and phrases. Prime geodesic theorem, Selberg zeta function, hyperbolic manifolds.

One should consult [9, Discussion 6.20., pp. 113-115 and Discussion 15.16., pp. 253-255] on the difficulties in improving Huber's $O\left(x^{\frac{3}{4}}(\log x)^{-\frac{1}{2}}\right)$. The best estimate for the error term in the prime geodesic theorem up to now is $O\left(x^{\frac{3}{4}}(\log x)^{-1}\right)$ obtained by Randol [20]. Its analogue has been also established for higher dimensional hyperbolic manifolds [2], improving Park's theorem [17, Th. 1.2.]. An important ingredient, implicitly [20] or explicitly [17], is the growth rate of the log-derivative of the Ruelle zeta vs. the Selberg zeta log-derivative [3].

Though the expected exponent on Riemann surfaces is $\frac{1}{2} + \varepsilon$, the above mentioned $\frac{3}{4}$ was successfully reduced only in the case of modular surfaces $\Gamma \setminus \mathcal{H}$, $\Gamma \subset PSL(2, \mathbb{Z})$. Iwaniec [12] obtained $\frac{35}{48} + \varepsilon$, Luo and Sarnak [15] $\frac{7}{10} + \varepsilon$, Cai [4] $\frac{71}{102} + \varepsilon$, Soundararajan and Young [22] $\frac{25}{36} + \varepsilon$.

Following Gallagher's [7] approach to the Riemann zeta, we shall prove that $\frac{7}{10}$ can be achieved for $\Gamma \subset PSL(2, \mathbb{R})$ outside a set of finite logarithmic measure.

2. MAIN RESULT

Let $\psi(x) = \sum_{N(P) \leq x} \Lambda(P) = \sum_{N(P) \leq x} \frac{\log N(P)}{1-N(P)^{-1}}$ and $\psi_1(x) = \int_1^x \psi(t) dt$ be the Chebyshev resp. integrated Chebyshev function, as usual. Recall that the error term $O\left(x^{\frac{3}{4}}(\log x)^{-1}\right)$ in the prime geodesic theorem corresponds to $O\left(x^{\frac{3}{4}}\right)$ in the explicit formula for ψ .

Our main result is given by the following theorem that substantially improves [14, Th. 1.]. and [1, Th. 2.]

Theorem. *Let $\Gamma \subset PSL(2, \mathbb{R})$ be a strictly hyperbolic Fuchsian group. There exists a set G of finite logarithmic measure such that*

$$\psi(x) = x + \sum_{\frac{7}{10} < \rho < 1} \frac{x^\rho}{\rho} + O\left(x^{\frac{7}{10}}(\log x)^{\frac{1}{5}}(\log \log x)^{\frac{1}{5}+\varepsilon}\right) \quad (x \rightarrow \infty, x \notin G),$$

where $\varepsilon > 0$ is arbitrarily small.

Proof. As the starting point, we shall take Hejhal's explicit formula for ψ_1 with an error term [9, Th. 6.16. on p. 110]

$$\begin{aligned} \psi_1(x) = & \alpha_0 x + \beta_0 x \log x + \alpha_1 + \beta_1 \log x + F\left(\frac{1}{x}\right) \\ & + \frac{x^2}{2} + \sum_{\substack{\rho \\ |\gamma| < T}} \frac{x^{\rho+1}}{\rho(\rho+1)} + O\left(\frac{x^2 \log x}{T}\right) \quad (x \rightarrow \infty), \end{aligned}$$

where $F(x) = (2g-2) \sum_{k=2}^{\infty} \frac{2k+1}{k(k-1)} x^{1-k}$.

The asymptotics of ψ is conveniently derived from the asymptotics of ψ_1 via the relation

$$\int_{x-h}^x f(t) dt \leq f(x)h \leq \int_x^{x+h} f(t) dt$$

valid for any non-decreasing function f , where $h > 0$.

We have

$$(1) \quad \begin{aligned} \psi(x) &\leq \frac{1}{h} \int_x^{x+h} \psi(t) dt = x + \sum_{\frac{1}{2} < \rho < 1} \frac{x^\rho}{\rho} + O(\log x) + O(h) \\ &+ \frac{1}{h} \left| \sum_{\substack{\operatorname{Re}(\rho)=\frac{1}{2} \\ |\gamma| \leq T}} \frac{(x+h)^{\rho+1} - x^{\rho+1}}{\rho(\rho+1)} \right| + O\left(\frac{x^2 \log x}{hT}\right). \end{aligned}$$

Now, for $Y < T$,

$$\sum_{\substack{\operatorname{Re}(\rho)=\frac{1}{2} \\ |\gamma| \leq T}} \frac{(x+h)^{\rho+1} - x^{\rho+1}}{\rho(\rho+1)} = \sum_{\substack{\operatorname{Re}(\rho)=\frac{1}{2} \\ |\gamma| \leq Y}} \frac{(x+h)^{\rho+1} - x^{\rho+1}}{\rho(\rho+1)} + \sum_{\substack{\operatorname{Re}(\rho)=\frac{1}{2} \\ Y < |\gamma| \leq T}} \frac{(x+h)^{\rho+1} - x^{\rho+1}}{\rho(\rho+1)}.$$

The trivial bound for the first sum on the right hand side is given by

$$(2) \quad \frac{1}{h} \left| \sum_{\substack{\operatorname{Re}(\rho)=\frac{1}{2} \\ |\gamma| \leq Y}} \frac{(x+h)^{\rho+1} - x^{\rho+1}}{\rho(\rho+1)} \right| = O\left(x^{\frac{1}{2}} \sum_{\substack{\operatorname{Re}(\rho)=\frac{1}{2} \\ |\gamma| \leq Y}} \frac{1}{|\rho|}\right) = O\left(x^{\frac{1}{2}} Y\right).$$

The second sum is split into

$$\sum_{\substack{\operatorname{Re}(\rho)=\frac{1}{2} \\ Y < |\gamma| \leq T}} \frac{(x+h)^{\rho+1}}{\rho(\rho+1)} - \sum_{\substack{\operatorname{Re}(\rho)=\frac{1}{2} \\ Y < |\gamma| \leq T}} \frac{x^{\rho+1}}{\rho(\rho+1)}.$$

Let

$$D_Y^T = \left\{ x \in [T, eT] : \left| \sum_{\substack{\operatorname{Re}(\rho)=\frac{1}{2} \\ Y < |\gamma| \leq T}} \frac{x^{\rho+1}}{\rho(\rho+1)} \right| > x^\alpha (\log x)^\beta (\log \log x)^\beta \right\}, \quad 1 < \alpha < \frac{3}{2}, \beta > 0.$$

Then,

$$\begin{aligned} \mu^\times D_Y^T &= \int_{D_Y^T} \frac{dt}{t} = \int_{D_Y^T} t^{2\alpha} (\log t)^{2\beta} (\log \log t)^{2\beta} \frac{dt}{t^{1+2\alpha} (\log t)^{2\beta} (\log \log t)^{2\beta}} \\ &\leq \frac{1}{(\log T)^{2\beta} (\log \log T)^{2\beta}} \int_{D_Y^T} \left| \sum_{\substack{\operatorname{Re}(\rho)=\frac{1}{2} \\ Y < |\gamma| \leq T}} \frac{t^{\rho+1}}{\rho(\rho+1)} \right|^2 \frac{t^{3-2\alpha}}{t^4} dt \\ &= O\left(\frac{T^{3-2\alpha}}{(\log T)^{2\beta} (\log \log T)^{2\beta}}\right) \int_T^{eT} \left| \sum_{\substack{\operatorname{Re}(\rho)=\frac{1}{2} \\ Y < |\gamma| \leq T}} \frac{t^{\rho+1}}{\rho(\rho+1)} \right|^2 \frac{dt}{t^4}. \end{aligned}$$

According to Koyama [14, p. 79],

$$\int_T^{eT} \left| \sum_{\substack{\operatorname{Re}(\rho)=\frac{1}{2} \\ Y < |\gamma| \leq T}} \frac{t^{\rho+1}}{\rho(\rho+1)} \right|^2 \frac{dt}{t^4} = O\left(\frac{1}{Y}\right).$$

Taking $Y = T^{3-2\alpha}(\log T)^{1-2\beta}(\log \log T)^{1-2\beta+\varepsilon}$, we obtain

$$\mu^\times D_Y^T \ll \frac{1}{\log T (\log \log T)^{1+\varepsilon}}.$$

For $n = \lfloor \log x \rfloor$, $T = e^n$, denote $E_n = D_Y^T$. Then, $\mu^\times E_n \ll \frac{1}{n(\log n)^{1+\varepsilon}}$ and $\mu^\times \cup E_n \ll \sum \frac{1}{n(\log n)^{1+\varepsilon}} < \infty$.

If $x \in [e^n, e^{n+1}) \setminus E_n$, one gets

$$(3) \quad \left| \sum_{\substack{\operatorname{Re}(\rho)=\frac{1}{2} \\ Y < |\gamma| \leq T}} \frac{x^{\rho+1}}{\rho(\rho+1)} \right| \leq x^\alpha (\log x)^\beta (\log \log x)^\beta.$$

We are interested in achieving $h < x^{\frac{3}{4}}$. If it happens that $x + h \in [e^n, e^{n+1}) \setminus E_n$, one shall have

$$\sum_{\substack{\operatorname{Re}(\rho)=\frac{1}{2} \\ Y < |\gamma| \leq T}} \frac{(x+h)^{\rho+1}}{\rho(\rho+1)} = O(x^\alpha (\log x)^\beta (\log \log x)^\beta)$$

as well, since $x + h < 2x$.

The other possibility is that $x + h \in [e^{n+1}, e^{n+2})$. In that case, we proceed as follows

$$\sum_{\substack{\operatorname{Re}(\rho)=\frac{1}{2} \\ Y < |\gamma| \leq T}} \frac{(x+h)^{\rho+1}}{\rho(\rho+1)} = \sum_{\substack{\operatorname{Re}(\rho)=\frac{1}{2} \\ Y < |\gamma| \leq eT}} \frac{(x+h)^{\rho+1}}{\rho(\rho+1)} - \sum_{\substack{\operatorname{Re}(\rho)=\frac{1}{2} \\ T < |\gamma| \leq eT}} \frac{(x+h)^{\rho+1}}{\rho(\rho+1)}.$$

For $x + h \in [e^{n+1}, e^{n+2}) \setminus E_{n+1}$, we get

$$(4) \quad \sum_{\substack{\operatorname{Re}(\rho)=\frac{1}{2} \\ Y < |\gamma| \leq eT}} \frac{(x+h)^{\rho+1}}{\rho(\rho+1)} = O(x^\alpha (\log x)^\beta (\log \log x)^\beta).$$

To estimate $\sum_{\substack{\operatorname{Re}(\rho)=\frac{1}{2} \\ T < |\gamma| \leq eT}} \frac{(x+h)^{\rho+1}}{\rho(\rho+1)}$, we shall consider

$$D_T^{\varepsilon T} = \left\{ x \in [eT, e^2 T) : \left| \sum_{\substack{\operatorname{Re}(\rho)=\frac{1}{2} \\ T < |\gamma| \leq eT}} \frac{x^{\rho+1}}{\rho(\rho+1)} \right| > x^\alpha (\log x)^\beta (\log \log x)^\beta \right\}.$$

By the argumentation above leading to the estimate of $\mu^\times D_Y^T$, we get

$$\mu^\times D_T^{eT} \ll \frac{T^{3-2\alpha}}{(\log T)^{2\beta} (\log \log T)^{2\beta}} \cdot \frac{1}{T} \ll \frac{1}{T^{2\alpha-2}}.$$

Recall that $n = \lfloor \log x \rfloor$, $T = e^n$ and denote $F_{n+1} = D_T^{eT}$. Notice that $\mu^\times F_{n+1} < \frac{1}{e^{(2\alpha-2)n}}$ and the series $\sum \frac{1}{e^{(2\alpha-2)n}}$ converges since $\alpha > 1$. Thus, if we additionally assume $x + h \in [e^{n+1}, e^{n+2}) \setminus F_{n+1}$, we get

$$(5) \quad \sum_{\substack{\operatorname{Re}(\rho) = \frac{1}{2} \\ T < |\gamma| \leq eT}} \frac{(x+h)^{\rho+1}}{\rho(\rho+1)} = O\left(x^\alpha (\log x)^\beta (\log \log x)^\beta\right).$$

Looking back at (1), and taking into account the relations (2), (3), (4) and (5), we are left to optimize

$$h, \frac{x \log x}{h}, x^{\frac{1}{2}} Y \text{ and } \frac{x^\alpha (\log x)^\beta (\log \log x)^\beta}{h}, \text{ i.e.,} \\ h, x^{\frac{1}{2}} \cdot x^{3-2\alpha} (\log x)^{1-2\beta} (\log \log x)^{1-2\beta} \text{ and } \frac{x^\alpha (\log x)^\beta (\log \log x)^\beta}{h}$$

since $\alpha > 1$, $T \approx x$ and $Y = O\left(x^{3-2\alpha} (\log x)^{1-2\beta} (\log \log x)^{1-2\beta+\varepsilon}\right)$.

Choosing $h \approx x^{\frac{\alpha}{2}} (\log x)^{\frac{\beta}{2}} (\log \log x)^{\frac{\beta}{2}}$, we get $\frac{1}{2} + 3 - 2\alpha = \frac{\alpha}{2}$ and $1 - 2\beta = \frac{\beta}{2}$. Hence, $\alpha = \frac{7}{5}$ and $\beta = \frac{2}{5}$. This completes the proof since the sets $E = \cup E_n$ and $F = \cup F_n$ have finite logarithmic measure.

□

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